Longstreth, M. D., Henry Hartshorne, M. D., J. Price Wetherill, Carl Seiler, M. D., William Goodell, M. D., Frank Thomson, Robert Patterson, Edward D. Cope, Charles S. Wurts, M. D.

Dinner was then served, and interesting addresses were delivered by Frederick Fraley, President; S. D. Gross, M. D., Hon. John Welsh, Robert E. Rogers, M. D., William Pepper, M. D., Eckley B. Coxe, and E. D. Cope, and at 10 o'clock P. M. the meeting adjourned.

Mr. Ashburner introduced the subject of a bill before Congress for establishing a Government Bureau of Mines.

On motion of Mr. Price, the consideration of the propriety of the Society's recommending to Government either the establishment of such a bureau, or the establishment of an executive department to take charge of the agricultural, mining and commercial interests of the nation, was referred to a committee consisting of the President, Mr. Fraley, as Chairman, Mr. Ashburner and Mr. Price.

And the meeting was adjourned.

On the Inclination of the Apparent to the True Horizon and the Errors rising thereof in Transit, Altitude, and Azimuth-Observations. By John Hagen, S. J., College of the Sacred Heart, Prairie Du Chien, Wisconsin.

(*Read before the American Philosophical Society, February 3, 1882.*)

In the year 1875, Mr. Hann, editor of the "Zeitschrift der Oesterreichischen Gesellschaft für Meteorlogie," called attention to a special kind of irregularities in the figure of the earth, which hitherto were not sufficiently taken into account. According to him *the most important perturbation of the ellipsoidal level of the sea arises from the continents attracting the waters of the surrounding oceans.* (See Mittheilungen der geogr. Gesellsch. zu Wien, N. 12, 1875.) He supports his statement by the fact, that the continents are to be compared to large mountains, which by necessity, must disturb the level of the sea in the same way, as the Cordilleras of South America, the Apennines in Italy and the Shehallien in Scotland were able to deviate the plumb-line, and again by the fact, that the force of gravity on islands was in average found greater than was forecast by calculation, from which Dr. Hann concludes that the level of the oceanic islands be lower than that of the shores of the continents. He estimates in general the vertical distance between the disturbed and the undisturbed level of the sea

to more than one thousand meters, and finally proposes the following problem to be solved :

To find such an Ellipsoid of Revolution, 1, as has the volume of the Earth; 2, that the sum of the Earth's elevations and depressions with regard to this Ellipsoid become a minimum.

This problem, however, as given by the author, seems to be indetermined, unless a third condition is added, viz.: that the rotation axis of the Ellipsoid is parallel to that of the Earth and their centres coincide.

Mr. Hann is of the opinion that the solution of this problem would afford the solution of another problem, open already a century ago, viz.: the answer to the question, why the meridian mensurations and the observations of the second's péndulum, made on different points of the surface of the Earth, afford such different values for the compression of the Earth? These observations, he says, ought to be reduced not to the actual level of the sea, but to the level of that regular ellipsoid to be found by the above problem, whose compression could then be found from these observations with greater accordance.

The treatise here published is intended not to solve Hann's problem, but to take one step farther towards its solution. This solution seems to be an impossibility as long as the inclination of the apparent towards the true horizon is not known, for as many places as possible, both as to magnitude and direction. On the following pages, therefore, the formulas shall be developed by which both the influence of this inclination on astronomical observations will be shown and the way suggested, how to determine its magnitude and direction. Astronomers are well aware of the influence that the deviation of the plumb-line exerts on finding the longitude and latitude of a place and have begun to distinguish between the *geodetic* and the *astronomical* position of a place. By the latter expression they mean the longitude and latitude of the apparent horizon; in other words, the *apparent* longitude and latitude of a place.* It is, however, evident, that for parallactic observations and especially for the transits of Venus and Mercury, not the apparent but the true longitude and latitude are needed. Consequently the following pages, though not giving direct means for finding the true position of an observatory, might be of some interest, as they at least call attention to the errors caused by the inclination of the horizon on astronomical observations.

Let the pole of the true or mathematical horizon be denoted by Z, and that of the apparent, or as we may call it, physical horizon by Z′, then the arc Z Z′ represents the inclination of the latter towards the former as to magnitude and direction. We resolve it into two rectangular components, one of which α may lie in the vertical plane of the instrument used, its positive direction being towards the "sight-line" of the observer, while the other component, β, may be positive right-hand of the observer. In case of an artificial horizon part of the inclination α may be caused by the

* NOTE.—About this distinction see Chauvenet's Manual of Spherical and Practical Astronomy, Vol. 1, Art. 86, 160, 213.

instrument and the piers on which it rests, hence, the distance of the artificial horizon varying with the zenith distance of the object observed, this part of the inclination α will be a function of the zenith distance, while the rest as well as the inclination β will be the same for the same azimuth. Now it will not be difficult to convince oneself *that the inclination α cannot influence but the observation of zenith distances and the inclination β but that of azimuths and hour-angles.* Nor is it difficult to foresee, that the inclination α will have a similar effect as the flexure of the telescope and graduated circle on account of their gravity, while the inclination β is comparable to the inclination of the horizontal rotation axis to the true horizon. The former two are functions of the zenith distance and may therefore be represented by periodic series, whose terms involve the sines and cosines of its multiples, while the latter two are merely functions of the azimuth.

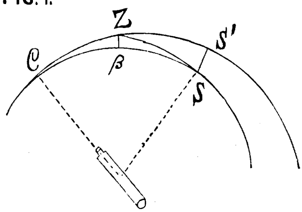
PART I.—*Influence of the inclination β on Azimuth- and Hour-angle Observations.*

We shall first suppose any altitude and azimuth instrument exactly adjusted so that the axis of collimation describes a great circle passing through the true zenith, and consider the influence exerted by the inclination of the artificial horizon on observations by reflection.

1. *Fundamental Formulas.*

If C denotes the point, in which the axis of collimation produced towards the eye-piece meets the celestial sphere, and Z the true zenith, the arc β will be perpendicular on the vertical plane C Z in the point Z. (Fig. 1.)

FIG. 1.



Again if through the end of the arc β and through C a great circle is put, the observed object S will be in this circle in the moment, when its reflected image passes over the middle thread of the telescope. From S let a perpendicular be drawn on the vertical plane of the instrument, which may be intersected in S^1, and let S and Z be joined by the arc of a great circle. Finally, let the small angles at Z and C be denoted respectively by $d\,A$ and C, and β be taken positively right-hand of the observer. Then we are not to forget, that $Z\,C = Z\,S$, *i. e.*, equal to the true zenith distance z of the observed object in the moment of observation. Now in the isosceles triangle S Z C we have

$$\cos z = \cot C \sin d\,A - \cos z \cos d\,A,$$

or introducing the angle β instead of C by the formula

$$\tan \beta = \tan C \sin z$$

we have

$$(1 + \cos d A) \text{ tant } \beta = \tan z \sin d A,$$

or simpler,

$$d A = 2 \beta \cot z, \tag{1}$$

which is the *correction of the azimuth,* for observations by reflection. There the azimuth is to be reckoned from south to west etc., and β right-hand of the observer.

The *correction of the hour-angle* may be derived from formula (1) by means of the well-known differential formula,

$$dt = \frac{\sin z}{\cos \delta \cos p} d A,$$

where p denotes the parallactic angle and δ the declination of the observed object. Thus we find

$$dt = 2 \beta \frac{\cos z}{\cos \delta \cos p} \tag{2}$$

For upper or lower culminations we have $\cos p = 1$, hence

$$dt = 2 \beta \frac{\cos z}{\cos \delta}. \tag{2^1}$$

For the sake of verification, this last formula may also be derived in the following way. Considering the great circle C Z S^1 as the meridian and joining S with the north pole N we have in the triangle S S^1 N

$$\sin dt = \frac{\sin S S^1}{\cos \delta}.$$

But in the triangle S S^1 C we have in like manner

$$\tan S S^1 = \tan C \sin 2 z,$$

since Z S^1 may be put equal to z and finally we have as above

$$\tan \beta = \tan C \sin z,$$

hence,

$$\tan S S^1 = \tan \beta \frac{\sin 2z}{\sin z} = 2 \tan \beta \cos z$$

and consequently by combining the first and last equation and supposing dt and β to be very small angles

$$dt = 2 \beta \frac{\cos z}{\cos \delta}. \tag{2^1}$$

2. *The azimuth instruments.*

The correction of the azimuth for the observation by reflection

$$d A = 2 \beta \cot z \tag{1}$$

has the meaning, that in such observations *the actual reading of the azimuth is by d A too small,* as long as β is positive right-hand of the observer.

If we now compare this correction with that for the inclination of the horizontal rotation axis to the true horizon we find both coincide except their constants. For if b denotes the elevation of the right-hand end of this axis above the true horizon, the correction of the azimuth is

$$d\,A = \mp\, b \cot z \begin{cases} - \text{ for direct image.} \\ + \text{ " reflected "} \end{cases}$$

as may be found in any Manual of Spherical Astronomy. Joining both corrections we have

$$\text{for direct image } d\,A = -\,b \cot z$$

$$\text{" reflect. " } d\,A = (2\beta + b) \cot z = -\,(b - d) \cot z.$$

if we put

$$d = 2\,(\beta + b) \qquad (3)$$

Hence the usual formula for correcting azimuth observations is to be modified for observations by reflection. For direct observations this formula is

$$a = A + \varDelta\,A - b \cot z - c \operatorname{cosec} z, \qquad (4)$$

where a denotes the absolute azimuth of the observed object, A the actual reading, $\varDelta\,A$ the index correction of the circle, so that $A + \varDelta\,A$ denotes the azimuth counted from the meridian point of the circle. b denotes as above the elevation of the right-hand end above the true horizon and $90^\circ + c$ is the angle formed by the axis of collimation with this same end. Hence for observations by reflection we have

$$a = A + \varDelta\,A - (b - d) \cot z - c \operatorname{cosec} z \qquad (5)$$

where z is not the reading of the vertical circle, but the zenith distance of the observed object. As we have defined the constant b as the inclination of the horizontal rotation axis to the *true* horizon, we, of course, cannot find it in the usual way with the striding level, this instrument being itself inclined to the true horizon by the unknown angle β. Hence we shall first find the constant $d = 2\,(\beta + b)$, which may be done in two ways, first by the striding level applied to the horizontal axis, which will give us

$$\beta + b = \tfrac{1}{2}\,d,$$

and secondly by observing the direct and reflected images of stars. Let θ be the sidereal time, when the direct image of a star passes over a certain azimuth and θ^1 the sidereal time, when the reflected image of the same star passes over the same azimuth, then we have the two equations

$$\text{direct image } a = A + \varDelta\,A - b \cot z - c \operatorname{cosec} z.$$

$$\text{reflect. " } a^1 = A + \varDelta\,A - (b - d) \cot z^1 - c \operatorname{cosec} z^1.$$

If now the observed star did not pass very near the zenith, we may neglect the two quantities

$$b\,(\cot z - \cot z^1) \text{ and } c\,(\operatorname{cosec} z - \operatorname{cosec} z^1)$$

as small of the second order and find by subtraction of the above equations

$$\frac{d}{2} = \beta + b = \frac{a^1 - a}{2} \tan z_0.$$

For z_0 may be taken the mean value of the two nearly equal zenith distances z and z^1, and if the instrument had no vertical circle, it may be computed from the declination, the latitude and the mean hour angle. Again we have

$$a^1 - a = \frac{d\,A}{dt}\,(\theta^1 - \theta),$$

where $\frac{d\,A}{dt}$ denotes the variation of the azimuth in the unit of time for the moment $\frac{1}{2}(\theta^1 + \theta)$.

Thus far it has been shown, how to find the value of d for one single azimuth, but it will be necessary to have the means of computing it for any azimuth. From the theory of the azimuth instruments it is known, that b is represented by the formula

$$b = i - i_0 \cos (A - A_0),$$

where i denotes the inclination of the horizontal axis to the azimuth circle, i_0 the inclination of this circle to the true horizon, while A is the azimuth of the observed object and A_0 a constant explained by the formula itself. The inclination β of the artificial horizon may be represented by a similar formula

$$\beta = - i_1 \sin (A - A_1), \qquad (6)$$

where i_1 is the constant deviation of the plumb line caused by local irregularities in the figure and density of the earth, A_1 the azimuth of its direction and A the azimuth of the observed object. Hence we find

$$\begin{aligned} \tfrac{1}{2}d = \beta + b &= i - i_0 \cos (A - A_0) - i_1 \sin (A - A_1) \\ &= i - \cos A\,(i_0 \cos A_0 - i_1 \sin A_1) - \sin A\,(i_0 \sin A_0 + i_1 \cos A_1) \end{aligned}$$

or if we put

$$\left.\begin{aligned} i_0 \cos A_0 - i_1 \sin A_1 &= i_2 \cos A_2 \\ i_0 \sin A_0 + i_1 \cos A_1 &= i_2 \sin A_2 \end{aligned}\right\} \qquad (7)$$

we find by a simple transformation

$$\tfrac{1}{2}\,d = \beta + b = i - i_2 \cos (A - A_2) \qquad (8)$$

To find the three constants i, i_2 and A_2 three observations are sufficient, which may be equally distributed in the usual way. Let d_1, d_2, d_3 be the values of d, corresponding to the three azimuths A, $A + 120^0$, $A + 240^0$ we find from (8)

$$\begin{aligned} \tfrac{1}{2}\,d_1 &= i - i_2 \cos (A - A_2) \\ \tfrac{1}{2}\,d_2 &= i + \tfrac{1}{2}\,i_2 \cos (A - A_2) + \tfrac{1}{2}\,i_2 \sin (A - A_2)\sqrt{3} \\ \tfrac{1}{2}\,d_3 &= i + \tfrac{1}{2}\,i_2 \cos (A - A_2) - \tfrac{1}{2}\,i_2 \sin (A - A_2)\sqrt{3} \end{aligned}$$

and by adding and subtracting these equations

$$\begin{aligned} i &= \tfrac{1}{6}\,(d_1 + d_2 + d_3) \\ i_2 \cos (A - A_2) &= \tfrac{1}{6}\,(d_2 + d_3 - 2d_1) \\ i_2 \sin (A - A_2) &= \frac{1}{2\sqrt{3}}\,(d_2 - d_3.) \end{aligned} \qquad (9)$$

If therefore either of the methods mentioned before, viz., by the striding

level or by observations of the direct and reflected image, is applied to three different azimuths, dividing the circle into three equal parts, the three constants i, i_2 and A_2 may be found by these formulas, and hence also the constant $\frac{1}{2}$d may be computed for any azimuth by the formula

$$\tfrac{1}{2}d = i - i_2 \cos (A - A_2) \qquad (8)$$

Thus we see, that b cannot be obtained in the usual way, before the collimation constant c has been found. But if the time is known, we may succeed in finding c in the following way: Let θ be the sidereal time, when the direct image passes over any azimuth, and θ^1 the time, when the same star passes over the same azimuth of the reversed instrument, then we have the two equations

$$a = A + \varDelta A - b \cot z - c \operatorname{cosec} z$$
$$a^1 = A + \varDelta A - b \cot z^1 + c \operatorname{cosec} z^1.$$

If again the star in the moment of observation did not pass very near the zenith, the quantity b $(\cot z - \cot z^1)$ may be neglected as small of the second order, hence we find by subtraction of the two equations

$$c = \tfrac{1}{2} (a^1 - a) \sin z_0$$

where z_0 is a mean value of z and z^1 and may be computed from the declination, the latitude and the mean hour-angle. Again we have

$$a^1 - a = \frac{d\,A}{dt} (\theta^1 - \theta)$$

where $\frac{d\,A}{dt}$ denotes the variation of the azimuth in the unit of time for the moment $\frac{1}{2}(\theta^1 + \theta)$.

If we now suppose the reading of the azimuth corrected as to the collimation constant, equation (4) becomes

$$a = A + \varDelta A - b \cot z. \qquad (4^1)$$

Again, if we observe the time of transit over the same azimuth for different stars, any two observations will afford an equation of this form.

$$b = \frac{a^1 - a}{\cot z - \cot z^1} = (a^1 - a) \frac{\sin z^1 \sin z}{\sin (z^1 - z)}.$$

The factor of $(a^1 - a)$ will turn out very small, consequently, b will be found with great exactness, if any star near the zenith is combined with any near the horizon. The quantities a and z may be computed from the hour-angle t by the formulas

$$\sin z \sin a = \cos \delta \sin t$$
$$\sin z \cos a = - \cos \varphi \sin \delta + \sin \varphi \cos \delta \cos t,$$

where δ denotes the declination of the star and φ the latitude of the place. The latter equation may be changed into the following form, more convenient for logarithmic computation :

$$\sin z \cos a = - m \cos (\varphi + M),$$

if we put

$$\sin \delta = m \cos M, \ \cos \delta \cos t = m \sin M.$$

If thus b is found for any azimuth, ΔA may be computed from (4^1). Yet b varies with the azimuth and is represented by the formula

$$b = i - i_0 \cos (A - A_0).$$

The constant i is already known from the equations (9) and hence it is enough to find b for any two azimuths in order to find i_0 and A_0. If we choose the two azimuths A and $A + 90^0$, we find

$$b_1 - i = - i_0 \cos (A - A_0)$$
$$b_2 - i = + i_0 \sin (A - A_0),$$

by which equations the two quantities i_0 and A_0 are fully determined. Thus we are able to compute b for any azimuth by the formula

$$b = i - i_0 \cos (A - A_0).$$

But from (7) we have the equations

$$\left.\begin{aligned} i_1 \sin A_1 &= + i_0 \cos A_0 - i_2 \cos A_2 \\ i_1 \cos A_1 &= - i_0 \cos A_0 + i_2 \sin A_2 \end{aligned}\right\} \qquad (10)$$

by which we finally find i_1 and A_1, *i. e.*, the constant inclination of the apparent to the true horizon, as far as it is caused by irregularities in the surface of the Earth, and the azimuth of its direction. This constant inclination i_1 however, is not yet the total inclination Z Z^1, since large instruments together with their piers may cause an inclination of the artificial horizon variable with the zenith distance of the observed object, as will be seen in Part II.

Finally, attention must be called to two things. First, if the observations mentioned above are made on different days, the positions of the stars are to be reduced to a common epoch, best to the beginning of the year. Secondly, though we have found the formulas for finding the constant inclination of the apparent to the true horizon as to magnitude and direction, we are not to forget, that these formulas suppose the perfect knowledge of the latitude and time of the place.

3. *The Transit instrument in the Meridian.*

The correction of the hour-angle for observations by reflection

$$dt = 2\,\beta \frac{\cos z}{\cos \delta} \qquad (2^1)$$

has the meaning, that in the moment, when the reflected image of any object passes over the middle thread of this instrument *its actual hour-angle is dt for upper transits and 180° + dt for lower transits,* if b is reckoned positive right-hand of the observer. Yet for these instruments the inclination β of the apparent horizon remaining always on the same side, it will be found more convenient to take β positive towards west and consequently to write the corrections for lower transits as follows:

$$dt = -2\,\beta \frac{\cos z}{\cos \delta}$$

while dt always denotes the increment of the hour-angle, which is reckoned in the usual way from south to west.

For upper culminations we have

$$z = + (\varphi - \delta) \text{ culmination south of the zenith}$$
$$z = - (\varphi - \delta) \quad \text{" north " "}$$

and for lower culminations $z = 180^\circ - (\varphi + \delta)$, hence the corrections for the hour-angle are

$$\text{for upper culm, } dt = 2\,\beta\,\frac{\cos(\varphi - \delta)}{\cos\delta}$$
$$\text{" lower " } dt = 2\,\beta\,\frac{\cos(\varphi + \delta)}{\cos\delta}.$$

If again we compare this correction with the one for the rotation axis not lying parallel to the horizon, we find them coincident, except the constant. For if b denotes the elevation of the west end of the rotation axis above the *true* horizon, we have the usual formula for upper culminations

$$dt = \mp b\,\frac{\cos(\varphi - \delta)}{\cos\delta} \begin{cases} - \text{ for direct image} \\ + \text{ " reflect. "} \end{cases}$$

and for lower culminations

$$dt = \mp b\,\frac{\cos(\varphi + \delta)}{\cos\delta} \begin{cases} - \text{ for direct image} \\ + \text{ " reflect. "} \end{cases}$$

where dt has the same meaning as above. Joining the two corrections and putting $2\,(\beta + b) = d$, as before, we find

$$\left.\begin{array}{ll} \multicolumn{2}{c}{\text{For upper culminations.}} \\ \text{direct image } dt = - & b\,\dfrac{\cos(\varphi - \delta)}{\cos\delta} \\ \text{reflect. " } dt = - & (b - d)\,\dfrac{\cos(\varphi - \delta)}{\cos\delta} \\ \multicolumn{2}{c}{\text{For lower culminations.}} \\ \text{direct image } dt = - & b\,\dfrac{\cos(\varphi + \delta)}{\cos\delta} \\ \text{reflect. " } dt = - & (b - d)\,\dfrac{\cos(\varphi + \delta)}{\cos\delta} \end{array}\right\} \quad (11)$$

where dt denotes the increment of the hour-angle. We need not consider separately the formulas for lower culmination, as we may deduce them from those for upper culmination at any time by simply substituting $180^\circ - \delta$ for δ.

In consequence of these considerations the formulas of Tobias Mayer, Bessel and Hansen are to be modified for observations by reflection as follows: Mayer's formula is the following

$$\tau = b\,\frac{\cos(\varphi + \delta)}{\cos\delta} + k\,\frac{\sin(\varphi - \delta)}{\cos\delta} + \frac{c}{\cos\delta}$$

where $\tau = -\,dt$ is the hour-angle east of the meridian, b the elevation of

the west end of the rotation axis above the true horizon, $90^\circ - k$ the azimuth of this west end and $90^\circ + c$ its angle with the line of collimation. Bessel's formula is

$$\tau = m + n \tan \delta + c \sec \delta,$$

and finally, Hansen's formula

$$\tau = b \sec \varphi + n (\tan \delta - \tan \varphi) + c \sec \delta,$$

where n denotes the declination of the west end of the rotation axis and $90^\circ - m$ its hour-angle. All these constants are in the following relations to each other:

$$\left.\begin{array}{ll} n = b \sin \varphi - k \cos \varphi & b = \quad n \sin \varphi + m \cos \varphi \\ m = b \cos \varphi + k \sin \varphi & k = - n \cos \varphi + m \sin \varphi \end{array}\right\} \quad (12)$$

For observations by *reflection* the constant b and consequently m and n are to be changed, say into b^1, m^1, n^1, by the following formulas:

$$\begin{array}{ll} b^1 = \quad -2\ \beta - b & = b - d \\ m^1 = m - 2 (\beta + b) \cos \varphi & = m - d \cos \varphi \\ n^1 = n - 2 (\beta + b) \sin \varphi & = n - d \sin \varphi. \end{array}$$

Hence the three formulas of Mayer, Bessel and Hansen become for observations by *reflection*,

$$\tau = (b - d) \frac{\cos (\varphi - \delta)}{\cos \delta} + k \frac{\sin (\varphi - \delta)}{\cos \delta} + \frac{c}{\cos \delta}$$

$$\tau = m + n \tan \delta + c \sec \delta - d \frac{\cos (\varphi - \delta)}{\cos \delta}$$

$$\tau = (b - d) \sec \varphi + (n - d \sin \varphi) (\tan \delta - \tan \varphi) + c \sec \delta.$$

As to determining the constants of these formulas, it will be seen, as in case of the azimuth instruments, that they cannot be found, unless the time of the place be known. First we will find the constant d, which may be done in two different ways, viz: by the striding level, which, being itself inclined to the true horizon by the angle β, cannot give the value of b, but it gives the value of

$$\beta + b = \tfrac{1}{2} d;$$

or by observing the transits of the direct and reflected image of a star. Let T and T^1 be the mean values of time for all the transits reduced to the middle thread for direct and reflected image, $\varDelta T$ the clock correction on sidereal time and a the star's apparent right ascension, then is evidently $a = T + \varDelta T + \tau$, hence

for direct image $a = T + \varDelta T \qquad + b \dfrac{\cos (\varphi - \delta)}{\cos \delta} + k \dfrac{\sin (\varphi - \delta)}{\cos \delta} + \dfrac{c}{\cos \delta}$

" reflect " $a = T^1 + \varDelta T + (b - d) \dfrac{\cos (\varphi - \delta)}{\cos \delta} + k \dfrac{\sin (\varphi - \delta)}{\cos \delta} + \dfrac{c}{\cos \delta}$

and by subtraction

$$\frac{d}{2} = \beta + b = \frac{T^1 - T}{2} \frac{\cos \delta}{\cos (\varphi - \delta)} \qquad (13)$$

which determination will be the more exact, the greater $\cos(\varphi - \delta)$, *i. e.* the nearer the observed star passed by the zenith.

The collimation constant is found in the usual way either by reversing the axis, or by using two horizontal collimating telescopes, and the constant n by observations of the upper and lower culmination. If then, we suppose the times of transit already corrected as to the errors arising from c and n, we find from Bessel's formula

$$\text{for direct image } a = T + \varDelta T + m$$

$$\text{" reflect. " } a = T^1 + \varDelta T + m - d \frac{\cos(\varphi - \delta)}{\cos \delta}$$

and from Hansen's formula

$$\text{for direct image } a = T + \varDelta T + b \sec \varphi$$

$$\text{" reflect. " } a = T^1 + \varDelta T + (b - d) \sec \varphi.$$

By these formulas it is made evident, that *neither m nor b can be found independently of the clock correction* But if this is known, Bessel's formula will give the constant m, or Hansen's formula b. The azimuth constant k may be determined by observations of upper and lower transits or be computed from (12). Thus, b being found, we may finally determine

$$\beta = \frac{d}{2} - b.$$

i. e. the west inclination of the apparent to the true horizon.

4. *The Transit Instrument in the Prime Vertical.*

From the general formula

$$dt = 2 \beta \frac{\cos z}{\cos \delta \cos p} \tag{2}$$

we shall obtain the formula for the transit instrument in the prime vertical by finding the value of cos p for the azimuth $A = 90^\circ$ and substituting it in the above formula. We have in general

$$\cos p \sin z = \cos \delta \sin \varphi - \sin \delta \cos \varphi \cos t.$$

But for the prime vertical we have the three special equations

$$\sin z = \cos \delta \sin t$$

$$\cos \delta = \frac{\cos \varphi \cos z}{\cos t}$$

$$\sin \delta = \sin \varphi \cos z.$$

Substituting these quantities successively into the three members of the general equation we find

$$\cos p \cos \delta = \sin \varphi \cos \varphi \cos z \tan t.$$

But from the three formulas for the prime vertical follows

$$\tan t = \frac{\tan z}{\cos \varphi}$$

consequently,

$$\cos p \cos \delta = \sin \varphi \sin z,$$

hence we have for observations by reflection with the transit instrument in the prime vertical the correction of the hour-angle.

$$dt = \frac{2\beta}{\tan z \sin \varphi} \tag{14}$$

The meaning of this correction is, that in the moment, when the reflected image of any object passes the middle thread of this instrument, *the actual hour-angle of the object observed is* $90^\circ + dt$ *or* $270^\circ + dt$, β being positive right-hand of the observer. Yet as also for this instrument the inclination β of the apparent horizon remains always on the same side, it will be found more convenient to take β positive towards north and consequently to write the correction of the hour-angle as follows:

$$dt = \pm \frac{2\beta}{\tan z \sin \varphi} \begin{cases} + \text{ Star west} \\ - \text{ " east.} \end{cases}$$

If we now compare this correction with the one for the rotation axis not lying parallel to the horizon, we find them coinciding except their constants. Let θ denote the sidereal time, when the star passed over the true prime vertical, and T the clock time, when it passed the middle thread of the instrument, and finally, Δ T the correction of the clock on sidereal time, then the theory of this instrument gives us these formulas for *direct* observations

$$\theta = T + \Delta T + \frac{b}{\tan z \sin \varphi} + \frac{k}{\sin \varphi} + \frac{c}{\sin z \sin \varphi} \quad \text{Star west}$$

$$\theta = T + \Delta T - \frac{b}{\tan z \sin \varphi} + \frac{k}{\sin \varphi} - \frac{c}{\sin z \sin \varphi} \quad \text{" east}$$

where b denotes the elevation of the north end of the rotation axis above the true horizon, $180^\circ - k$ the azimuth of this same end, and $90^\circ + c$ its angle with the sight-line of the telescope. For observations by reflection, $180^\circ - z$ is to be substituted for z, which changes only the sign of b. But besides this, the artificial horizon being inclined to the north, the reflected image will be observed after the star passed over the prime vertical in the west and before it passed over the same in the east. Hence, if we put $d = 2(\beta + b)$ as before, the first fraction of the above equations becomes

$$-\frac{b + 2\beta}{\tan z \sin \varphi} = + \frac{b - d}{\tan z \sin \varphi} \quad \text{Star west}$$

$$+\frac{b + 2\beta}{\tan z \sin \varphi} = - \frac{b - d}{\tan z \sin \varphi} \quad \text{" east.}$$

Hence the two formulas for the transit instrument in the prime vertical are to be modified for *observations by reflection* in the following way:

$$\theta = T + \Delta T + \frac{b - d}{\tan z \sin \varphi} + \frac{k}{\sin \varphi} + \frac{c}{\sin z \sin \varphi} \quad \text{Star west}$$

$$\theta = T + \Delta T - \frac{b - d}{\tan z \sin \varphi} + \frac{k}{\sin \varphi} - \frac{c}{\sin z \sin \varphi} \quad \text{" east.}$$

Also in this case we shall see, that the constants cannot be found without

the time and latitude of the place being known. First d may be determined, as in former cases, either by the striding level, which will give the angle

$$\tfrac{1}{2}\,d = b + \beta,$$

or by observing the direct and reflected image of a star either in west or in east. By subtracting the two corresponding equations we find

$$\frac{d}{2} = \beta + b = \frac{T^1 - T}{2} \tan z \sin \varphi,$$

where stars are to be chosen, that pass near the zenith, The collimation constant c may be determined by reversing the axis and observing in both cases the time of transit. As in this case the sign of c alone is changed, we find by subtracting the two corresponding equations

$$c = \frac{T^1 - T}{2} \sin z \sin \varphi,$$

where stars passing near the zenith are again preferable. Both operations may be performed by first observing the transits over some threads and then, after having moved the instrument, over the rest, and by reducing them to the middle thread, or if the observations are taken on different days, the rate of the clock must be known and added to the observed time.

Let us now suppose the time T being already corrected as to the collimation, then by observing the same star east and west we may find both constants b and k. In this case the equations are

$$\theta = T + \varDelta T + \frac{b}{\tan z \sin \varphi} + \frac{k}{\sin \varphi} \text{ Star west,}$$

$$\theta^1 = T^1 + \varDelta T - \frac{b}{\tan z \sin \varphi} + \frac{k}{\sin \varphi} \text{ `` east.}$$

By subtracting we have

$$b = \tan z \sin \varphi \left[\, \tfrac{1}{2}(\theta - \theta^1) - \tfrac{1}{2}(T - T^1)\right].$$

Should the clock corrections not be the same T^1 were to be corrected by the rate. Now $\frac{1}{2}(\theta - \theta^1) = t$ is the hour-angle of the star in the moment when it passes over the true prime vertical and may be computed from the latitude of the place and the star's declination by the formula

$$\cos t = \frac{\tan \delta}{\tan \varphi}$$

or better still from the formula

$$\tan \tfrac{1}{2}\, t^2 = \frac{\sin(\varphi - \delta)}{\sin(\varphi + \delta)}\,.$$

The errors in the observation of $T - T^1$ will also here be the smaller, the smaller tan z, *i. e.* the nearer the star passes the zenith. Now d and b being known we find the north inclination of the apparent horizon

$$\beta = \tfrac{1}{2}\, d - b.$$

By adding the above equations we find

$$k = \sin \varphi \left[\tfrac{1}{2}(\theta + \theta^1) - \tfrac{1}{2}(T + T^1) - \Delta T\right],$$

or as $\frac{1}{2}(\theta + \theta^1) = a$ is the star's right ascension

$$k = \sin \varphi \left[a - \tfrac{1}{2}(T + T^1) - \Delta T\right].$$

Part II.—*Influence of the inclination α on Altitude Observations.*

By α we have denoted that component of the inclination $Z\,Z^1$ of the apparent to the true horizon, which lies in the vertical plane of the instrument used. With large instruments part of this component may be caused by the instrument and its piers, and is, therefore, as was explained in the beginning, depending on the zenith distance of the object observed. The other part of α is according to former notations [see formula (6)]

$$q = i_1 \cos (A - A_1) \qquad (15)$$

and is caused by the constant local irregularities in the figure and density of the earth. The first part of α will have an effect on altitude observations quite analogous to the flexure of the instrument. This latter correction is generally represented by the series

$$\begin{aligned} &a^1 \cos z + a^{11} \cos 2\,z + a^{111} \cos 3\,z + \ldots \\ &+ b^1 \sin z + b^{11} \sin 2\,z + b^{111} \sin 3\,z + \ldots \end{aligned}$$

and its sign is understood so, that if z is the reading of the zenith distance of a star

$$z + a^1 \cos z + \ldots \quad + b^1 \sin z + \ldots.$$

represents the true zenith distance freed from flexure. If for instance N denotes the reading of the Nadir point (for which $z = 180^\circ$,)

$$N - a^1 + a^{11} - a^{111} + \ldots$$

will represent the true nadir freed from flexure.

By a similar formula the component α may be represented this way

$$\left.\begin{aligned} \alpha = q &+ a_1{}^1 \cos z + a_1{}^{11} \cos 2\,z + a_1{}^{111} \cos 3\,z + \ldots \\ &+ b_1{}^1 \sin z + b_1{}^{11} \sin 2\,z + b_1{}^{111} \sin 3\,z + \ldots \end{aligned}\right\} \qquad (16)$$

For the nadir ($z = 180^\circ$) we have

$$\alpha_0 = q - a_1{}^1 + a_1{}^{11} - a_1{}^{111} + \ldots$$

Now let z denote the reading of the instrument, ζ the true zenith distance of the object S observed, and N the reading of the nadir, then we shall have for direct observations (Fig. 2).

$$\begin{aligned} &z + a^1 \cos z + a^{11} \cos 2\,z + a^{111} \cos 3\,z + \ldots \\ &\quad + b^1 \sin z + b^{11} \sin 2\,z + b^{111} \sin 3\,z + \ldots \\ &- (N + 180^\circ - a^1 + a^{11} - a^{111} + \ldots) + \alpha_0 = \zeta \end{aligned}$$

Again let z^1 be the reading of an observation by reflection and we shall have

$$\begin{aligned} &z^1 - a^1 \cos z + a^{11} \cos 2\,z - a^{111} \cos 3\,z + \ldots \\ &\quad + b^1 \sin z - b^{11} \sin 2\,z + b^{111} \sin 3\,z - \ldots \\ &\quad - (N + 180^\circ - a^1 + a^{11} - a^{111} + \ldots) + \alpha_0 = 180^\circ - \zeta + 2\,\alpha \end{aligned}$$

Let now the rotation axis of the instrument be reversed so that the graduation runs in the contrary direction and z^{II} be the reading of a direct observation and we shall have

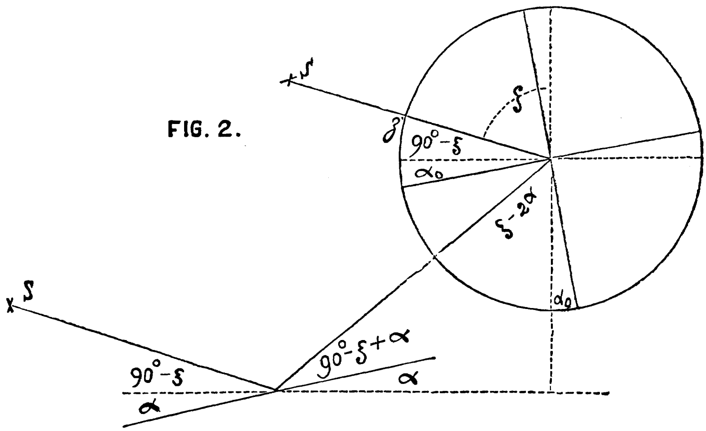
$$\begin{aligned} z^{II} &+ a^{I} \cos z + a^{II} \cos 2z + a^{III} \cos 3z + \ldots \\ &- b^{I} \sin z - b^{II} \sin 2z - b^{III} \sin 3z - \ldots \\ &- (N + 180^\circ - a^{I} + a^{II} - a^{III} + \ldots) - \alpha_0 = 360^\circ - \zeta \end{aligned}$$

Let finally z^{III} be the reading of an observation by reflection in the same position of the instrument, and we shall have

$$\begin{aligned} z^{III} &- a^{I} \cos z + a^{II} \cos 2z - a^{III} \cos 3z + \ldots \\ &- b \sin z + b^{II} \sin 2z - b^{III} \sin 3z + \ldots \\ &- (N + 180^\circ - a^{I} + a^{II} - a^{III} + \ldots) - \alpha_0 = 180^\circ + \zeta - 2\alpha \end{aligned}$$

But from the explanations in the first part, it is evident, that with obser-

FIG. 2.



vations by reflection a star is observed out of the vertical plane of the instrument, so that the azimuth of the star is by

$$d\,A = 2\,\beta \cot z$$

greater than the azimuth of the reading. Hence, if we want to compare with each other the four equations given above, we are to reduce all the zenith distances to the same azimuth. This may be effected by the well-known formula

$$dz = \tan p \sin z\, d\,A,$$

which by substituting the above value of $d\,A$ becomes

$$dz = 2\,\beta \tan p \cos z. \qquad (17)$$

Here, as in Part I, p denotes the parallactic angle. The meaning of formula (17) is not, as if the inclination β of the artificial horizon could prevent the observer from reading the actual zenith distance of the star, it means that the actual zenith distance is by dz greater, than it would be, if the star were still in the azimuth of the instrument.

Hence, with the two observations by reflection mentioned above, the readings z^1 and z^{111} are to be diminished by $2\beta \tan p \cos z$, in order to have in all the four equations the same true zenith distance belonging to the same azimuth. If the observation by reflection is taken in the meridian, where tan p is very small, this correction may be omitted as small of the second order. The same value of dz may also be found by the usual differential formula

$$dz = \cos\delta \sin p\, dt$$

and the following formula, which was developed above

$$dt = 2\beta \frac{\cos z}{\cos\delta \cos p}.$$

If for brevity's sake we denote the apparent zenith point, corrected as to flexure, by Z_1 and put

$$Z_1 = 180^\circ + N - a^1 + a^{11} - a^{111} + \ldots$$

our four equations mentioned several times will become

$$\left.\begin{aligned}
\zeta &= z + a^1 \cos z + a^{11} \cos 2z + a^{111} \cos 3z + \ldots \\
&\quad + b^1 \sin z + b^{11} \sin 2z + b^{111} \sin 3z + \ldots \\
&\quad - Z_1 + \alpha_0. \\
180^\circ - \zeta &= z^1 - (a^1 - 2a_1^{\,1}) \cos z + (a^{11} - 2a_1^{\,11}) \cos 2z - \ldots \\
&\quad + (b^1 - 2b_1^{\,1}) \sin z - (b^{11} - 2b_1^{\,11}) \sin 2z + \ldots \\
&\quad - Z_1 - 2q + \alpha_0 - 2\beta \tan p \cos z. \\
360^\circ - \zeta &= z^{11} + a^1 \cos z + a^{11} \cos 2z + a^{111} \cos 3z + \ldots \\
&\quad - b^1 \sin z - b^{11} \sin 2z - b^{111} \sin 3z - \ldots \\
&\quad - Z_1 - \alpha_0. \\
180^\circ + \zeta &= z^{111} - (a^1 + 2a_1^{\,1}) \cos z + (a^{11} + 2a_1^{\,11}) \cos 2z - \ldots \\
&\quad - (b^1 + 2b_1^{\,1}) \sin z + (b^{11} + 2b_1^{\,11}) \sin 2z - \ldots \\
&\quad - Z_1 + 2q - \alpha_0 - 2\beta \tan p \cos z.
\end{aligned}\right\} \quad (18)$$

These equation are sufficient to find the probable values of the constants a, b, a_1 and b_1 by observations of different stars. The constants a however can be eliminated, so that, to find zenith distances, we need not know but the constants b and q. For we find

$$\zeta - 180^\circ = \tfrac{1}{2}(z - z^{11}) + b^1 \sin z + b^{11} \sin 2z + b^{111} \sin 3z + \ldots + \alpha_0 \quad (19)$$

The b being found by this equation, the constants a, may be found by the following one

$$\begin{aligned}
-\zeta = \tfrac{1}{2}(z^1 - z^{111}) &+ 2a_1^{\,1} \cos z - 2a_1^{\,11} \cos 2z + \ldots \\
&+ b^1 \sin z - b^{11} \sin 2z + \ldots - 2q + \alpha_0.
\end{aligned}$$

The constants a may be determined from

$$180^\circ = \tfrac{1}{2}(z + z^{11}) + a^1 \cos z + a^{11} \cos 2z + \ldots - Z_1$$

and afterwards also the b_1 from

$$\begin{aligned}
180^\circ = \tfrac{1}{2}(z^1 + z^{111}) &- a^1 \cos z + a^{11} \cos 2z - \ldots \\
&- 2b_1^{\,1} \sin z + 2b_1^{\,11} \sin 2z - \ldots - Z_1 - 2\beta \tan p \cos z.
\end{aligned}$$

The equations (18) and all the others developed from them show, that

the true zenith distance ζ cannot be separated from the constant a_0 or, to speak more exactly, from the constant q, they giving always the value of ζ — q. Nor will it be possible to separate zenith distances from this inclination by *sextants* or *reflecting circles*. The inclination β perpendicular to the plane of the sextant or reflecting circle has indeed no influence on finding altitudes, yet this is the case with the inclination α in the plane of the instrument, all the readings of altitudes being too great by the angle α, if an artificial horizon is used, while in case of a sea horizon the dip will be affected by this inclination. Neither of these errors can be eliminated by these instruments. Thus by altitude observations the inclination of the artificial horizon may be found *as far as it depends on the attraction of the instrument and its piers*, but not as far as it depends on local irregularities of the earth.

Now to come to a conclusion, the question turns up to the astronomer, by what means he will find the latitude and the time of his place. Since in case that his apparent meridian line is not parallel to the true horizon, all observations of stars will give him the latitude not of his place, but of such places, whose true horizon is parallel to his apparent meridian line. And in like manner if the plane of his apparent meridian does not go through the centre of the earth, all observations of stars will furnish him with the time not of his place, but of such places as are lying in a plane parallel to his apparent meridian and touching the centre of the earth. Consequently, all the methods of finding the longitude by immediate transportation of time or by observation of signals *visible at the same instant* will give him the longitude not of his place, but of the places just defined.

He must therefore look out for other means to find the errors in the determination of the latitude and the longitude of his place, and consequently also the constants of correction for his instruments, and such means seem to be *geodetic mensurations* and the observation of *parallactic phenomena*. If as many places of the earth as possible are combined by such observations and mensurations and the condition is made, that the sum of the squares of differences between the calculated and observed longitudes and latitudes becomes a minimum, the probable errors in determining the position of these places may be found. The first method has been partially employed by Prof. Schmidt in Göttingen and later also by the U. S. Coast Survey.* On the instigation of the celebrated Gauss Prof. Schmidt made use of the different meridian mensurations to calculate the dimensions of the terrestrial ellipsoid, so that the sum of the squares of differences between the computed and observed latitudes was a minimum. He found for the mean error of latitudes 3″ .193. But it may be interesting to have the complete result of his computation here reprinted from his "Lehrbuch der mathem. u. phys. Geography, Göttingen, 1829, I. p. 199."

* Report for 1853.

Tarqui	3°	4′	30″.83 + 1″.87
Cotchesqui	0	2	37.83 — 1.87
Trivandeporum	11	44	52.59 — 0.58
Paudree	13	19	49.02 + 0.57
Punnæ	8	9	38.39 — 1.78
Putchapolliam	10	59	48.93 — 1.22
Dodagoontah	12	59	59.91 + 3.54
Namthabad	15	6	0.64 — 0.54
Formetera	38	39	56.11 + 3.40
Montjouy	41	21	45.45 + 2.55
Barcelona	41	22	47.16 + 0.82
Perpignan	42	41	58.01 — 4.16
Carcassone	43	12	54.31 — 1.02
Evaux	46	10	42.19 — 5.88
Pantheon	48	50	48.94 + 0.37
Dünkirchen	51	2	8.74 + 3.92
Göttingen	51	31	47.85 — 2.76
Altona,	53	32	45.27 + 2.76
Dunnose	50	37	8.21 — 1.86
Greenwich	51	28	40.00 + 0.94
Blenheim	51	50	27.09 + 3.01
Arburyhill	52	13	28.19 + 1.83
Clifton	53	27	31.99 — 3.91
Mallörn	65	31	31.06 + 1.31
Pahtawara	67	8	51.41 — 1.31

In like manner also mensurations of Parallels might serve to find the errors in longitude. Amongst the *parallactic* phenomena, which may contribute towards finding the errors in longitude and latitude, especially solar eclipses and occultations of stars are to be mentioned. If in the equation, which represents the condition of a certain place of the earth lying in the surface of the cone of shadow, not only the longitude, but also the latitude and sidereal time, are supposed to be erroneous,* very likely part of the errors, for which formerly the ephemerides were made responsible, must be ascribed to the inclination of the apparent horizon. Thus longitude and latitude of an Observatory being approximately corrected by any of these methods, the formulas given in the preceding pages will furnish the means of finding the constants of correction for the instruments, and finally also *the inclination of the apparent to the true horizon as to magnitude and direction.*

* Brünnow in his "Lehrbuch der Sphärischen Astronomie," p. 329, develops this equation, supposing only the Ephemerides to be erroneous, Chauvenet in his "Manual of Spherical and Practical Astronomy," 5th ed. vol. i, p. 523, regards the corrections of the coördinates of the place of observation as depending only upon the correction of the eccentrictity of the terrestrial meridian, supposing the latitude itself as well as the sidereal time to be correct.